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Model-completeness and decidability of the additive structure of integers expanded with a function for a Beatty sequence. (English. English summary)

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The (homogeneous) *Beatty sequence* associated with an irrational parameter α is the integer sequence $\lfloor n\alpha\rfloor$. There are a number of natural structures extending the additive group $(\mathbb{Z},+)$ which can be associated to such a sequence. The one considered here, denoted $\mathcal{Z}_\alpha$, adds the function $\lfloor n\alpha\rfloor$ with n ranging over $\mathbb{Z}$ (to avoid introducing the order as well). A weaker structure, which we may call $\mathcal{Z}'_\alpha$, will add only the range of this function.

Somewhat richer structures have also been studied, notably the structure $\mathcal{R}_\alpha$ obtained by enriching $(\mathbb{R},<)$ by predicates for both $\mathbb{Z}$ and $\mathbb{Z}\alpha$. In the very particular case in which α is a quadratic irrational, it has been shown that the first-order theory of $\mathcal{R}_\alpha$ can be analyzed using automata theory. This depends in a very direct way on the fact that the continued fraction expansion is periodic. Furthermore, in the case in which α is the golden ratio, the structure $\mathcal{S}_\alpha$ which includes the multiplication map by α as well can be similarly understood [P. Hieronymi, J. Symb. Log. **81** (2016), no. 3, 1007–1027; MR3569117].

At the other end of the spectrum, the structure $\mathcal{Z}'_\alpha$ has been shown to be quite tame [A. Günaydın and M. Özsahakyan, Ann. Pure Appl. Logic **173** (2022), no. 3, Paper No. 103062; MR4345246]. In the present paper a reasonably explicit set of axioms for the complete (and model complete) theory of the structure $\mathcal{Z}_\alpha$ is given for the case in which α is transcendental, via a quantifier elimination procedure. For this axiomatization to be completely explicit, so that the corresponding theory is decidable, the parameter α should itself be computable.

The idea is to interpret first-order formulas as "talking about" residues mod 1 rather than integer parts, in an informal sense, and to use general results on equidistribution. In particular, it is noticed that from this point of view, there is a natural first-order definition of a *dense* linear order on $\mathbb{Z}$.

The extension of these results to the case of algebraic α is sketched in the final section, §5, along with a number of other related results and questions—notably, the question of the effect of adding the usual order on $\mathbb{Z}$ to the language of the structure $\mathcal{Z}_\alpha$.

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[References]

1. J.R. Büchi, On a decision method in restricted second order arithmetic, in: Logic, Methodology and Philosophy of Science, Proc. 1960 Internat. Congr., Stanford Univ. Press, Stanford, Calif., 1962, pp. 1–11. MR0183636
2. G. Conant, There are no intermediate structures between the group of integers and Presburger arithmetic, J. Symb. Log. 83 (1) (2018) 187–207. MR3796282
3. G. Conant, A. Pillay, Stable groups and expansions of $(\mathbb{Z},+,0)$, Fundam. Math. 242 (3) (2018) 267–279. MR3826645
4. I.G. Connell, Some properties of Beatty sequences II, Can. Math. Bull. 3 (1) (1960) 17–22. MR0110683
5. B. Green, T. Tao, The primes contain arbitrarily long arithmetic progressions, Ann. Math. 167 (2) (2008) 481–547. MR2415379
6. A. Günaydin, M. Özsahakyan, Expansions of the group of integers by Beatty sequences, Ann. Pure Appl. Log. 173 (3) (2022) 103062. MR4345246
7. G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, sixth ed.,

Oxford University Press, Oxford, 2008. MR2445243

8. P. Hieronymi, Expansions of the ordered additive group of real numbers by two discrete subgroups, J. Symb. Log. 81 (3) (2016) 1007–1027. MR3569117
9. P. Hieronymi, D. Ma, R. Oei, L. Schaeffer, C. Schulz, J. Shallit, Decidability for Sturmian words, arXiv:2102.08207 [abs], 2021. MR4405042
10. I. Kaplan, S. Shelah, Decidability and classification of the theory of integers with primes, J. Symb. Log. 82 (3) (2017) 1041–1050. MR3694340
11. M. Khani, A. Zarei, The additive structure of integers with the lower Wythoff sequence, Arch. Math. Log. (2022) 1–13, https://doi.org/10.1007/s00153-022-00846-2. MR4535932
12. T. Tao, Structure and Randomness: Pages from Year One of a Mathematical Blog, American Mathematical Society, 2008. MR2459552

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.