Non-standard Fibonacci numbers and a model-complete axiomatization for the expansion of $\langle \mathbb{N}, +, < \rangle$ with a Beatty sequence

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Definability

Information hidden within a structure

Hidden information: Definability

- Order is hidden information in $\langle \mathbb{R}, +, \cdot \rangle$: let $a, b \in \mathbb{R}$, then

$$a < b \text{ iff } \langle \mathbb{R}, +, \cdot \rangle \models \exists z \quad b = a + z^2.$$

- Order is hidden information in $\langle \mathbb{Z}, +, \cdot \rangle$, for a completely different reason: let $a, b \in \mathbb{Z}$, then

$$a < b \text{ iff } \langle \mathbb{Z}, +, \cdot \rangle \models \exists z_1, z_2, z_3, z_4 \quad b = a + z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

Definition

Let M be a structure. A set $X \subseteq M^n$ is definable if there is a formula φ such that $X = \{\bar{a} \in M : \varphi(\bar{a})\}$.

The hidden information may be surprising!

- Using only $+, \cdot$ can you give a formula φ to express being an integer in $\langle \mathbb{Q}, +, \cdot \rangle$:

$$\mathbb{Z} = \{a \in \mathbb{Q} : \langle \mathbb{Q}, +, \cdot \rangle \models \varphi(a)\}$$

DEFINABILITY AND DECISION PROBLEMS IN ARITHMETIC 107

$$\bigwedge_{A,B} (\{(\bigvee_{X,Y,Z} 2 + BZ^2 = X^2 + AY^2) \wedge \bigwedge_{M} [(\bigvee_{X,Y,Z} 2 + ABM^2 + BZ^2 = X^2 + AY^2) \rightarrow (\bigvee_{X,Y,Z} 2 + AB(M+1)^2 + BZ^2 = X^2 + AY^2)]\} \rightarrow (\bigvee_{X,Y,Z} 2 + ABN^2 + BZ^2 = X^2 + AY^2)).$$

Fact (Kleene)

Given $\langle \mathbb{N}, +, \cdot \rangle$ it is possible to write a formula φ that expresses the following:

> *The Turing machine number n halts at input m after k steps.*

Decidability

Is there is an algorithm?

Decidability

Definition

T is decidable if there is an algorithm that gets φ as input and tells you whether $T \vdash \varphi$ or $T \nvdash \varphi$.

Remark (separate but important story)

$T \nvdash \varphi$ is **not** the same thing as $T \vdash \neg\varphi$.

Two extreme ends about natural numbers

- Gödel's incompleteness: $\langle \mathbb{Z}, +, \cdot \rangle$ is undecidable. (see Fact on page: 4)
- Presburger arithmetic: $\langle \mathbb{N}, +, < \rangle$ is decidable.

Question

How much multiplication is required for Gödel phenomenon?

Question

Is undecidability equivalent to the occurrence of Gödel phenomenon?

Model-Completeness

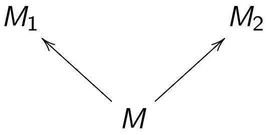
Can you solve systems of equations?

Some more elementary logic

Consider a system of equations:

$$\Sigma = \begin{cases} \psi_1(x, \bar{a}_1) \\ \vdots \\ \psi_n(x, \bar{a}_n) \end{cases}$$

- Quantifier-elimination: with coefficients from M, if solvable in M_1 solvable in M_2.



- Model-completeness: with coefficients from M_1 if solvable in M_2 solvable in M_1.

$$M_1 \longrightarrow M_2$$

Logically, quantifier-elimination means that every formula has a quantifier-free equivalent, and model completeness means that every formula has an equivalent in existential form.

Relation between the themes

How do we prove that a certain theory is decidable?

- Find the algorithm!
- Interpret it somewhere decidable, or
- Use model theory!

Definition (Complete theory)

T is complete if for any sentence φ either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Theorem

Recursive complete $\Rightarrow$ *decidable.*

Corollary

Model-complete+has prime model+recursive$\Rightarrow$ *decidable.*

The Talk!

Hieronymi: Regarding the question on page 7, consider traces of multiplication

Theorem (Hieronymi, link)

- $\langle \mathbb{R}, \mathbb{Z}, +, \alpha \cdot x, < \rangle$ *is decidable if and only if* α *is a square, say the golden ratio (proof: it can be interpreted— in a somehow complicated way— in the Büchi structure* $(\mathbb{N}, P(\mathbb{N}), \in, s.)$
- $\langle \mathbb{R}, \mathbb{Z}, +, \alpha\mathbb{Z}, \beta\mathbb{Z}, < \rangle$ *with two traces of multiplication is undecidable, because it interprets multiplication. (here* α *and* β*, 1 are independent over* $\mathbb{Q}$*).*

Question (Hieronymi)

Can you prove it model-theoretically?

Theorem (Khani, Valizadeh, Zarei)

- $\langle \mathbb{Z}, +, -, 0, 1, f(x) = \lfloor \varphi x \rfloor \rangle$ *is decidable (link)*
- $\langle \mathbb{Z}, +, -, 0, 1, f(x) = \lfloor e \cdot x \rfloor \rangle$ *is decidable, (link)*
- $\langle \mathbb{Z}, +, -, 0, 1, <, f(x) = \lfloor \varphi x \rfloor \rangle <$ *is decidable, submitted (link)*

φ is the golden ratio and e is the Euler number

Theorem for this talk:

Theorem (Khani, Valizadeh, Zarei)

$\langle \mathbb{Z}, +, -, 0, 1, <, f(x) = \lfloor \varphi x \rfloor \rangle$ *is model-complete and axiomatizable with a recursive theory. Hence it is decidable.*

Observation

Being a first order strucutre, there is no decimal number in $\langle \mathbb{Z}, +, -, 0, 1, <, f(x) = \lfloor \varphi x \rfloor \rangle$. But it possible to *talk about* the decimal parts:

$$[\varphi x] < [\varphi y] \leftrightarrow f(y - x) = f(y) - f(x) \quad \text{(order property)}$$

Axiom

$[\varphi x] < [\varphi y]$ is a linear order.

Theorem

$[\varphi x] < [\varphi y]$ *is actually a* ***dense*** *linear order.*

Proof.

$[\varphi x] < [x + \bar{f}(y - x)] < [\varphi y]$, $\bar{f} = f + id$. □

Kronecker's Lemma

Let r be an irrational number and consider the sequence $(rn)_{n\in\mathbb{N}}$. Then for each subinterval $I \subseteq [0,1]$ there is $n \in \mathbb{N}$ such that $[rn] \in I$.

Interesting:

In our previous work, we put Kronecker's theorem as an axiom, but according to the previous theorem, it is already a theorem in our theory. Indeed this is the main reason that $\langle \mathbb{Z}, +, -, 0, 1, f(x) = \lfloor \varphi x \rfloor \rangle$ admits quantifier elimination (see KZ).

Some important issue to address regarding model-completeness

Question

Let $M_1 \subseteq M_2$ be two models (of your appropriate theory for this structure). Let $a, b, c, d \in M_1$. Assume that M_2 models the following sentence:

$$\exists x \begin{cases} a < x < b \\ [\varphi c] < [\varphi x] < [\varphi d] \end{cases}$$

Is there any reason to think that the formula above also holds in M_1?

Remember

The definition of model-completeness from page 9.

Minimum of the decimal parts in an interval

Axiom

Let M be a model. In any interval $[a, b]$ there is c such that

$$[\varphi c] = \min\{[\varphi x] : x \in [a, b]\}.$$

Call this: $c^M_{[a,b]}$. If the answer to Question 4 is positive, then we will have:

$$c^{M_2}_{[a,b]} = c^{M_1}_{[a,b]}.$$

Because otherwise there would exist an element smaller than $c^{M_1}_{[a,b]}$ in M_2.

Restrict to intervals $[0, a]$

Theorem

$c^{\mathbb{N}}(0, n)$ is the largest Fibonacci number smaller than n, with an even index.

Corollary

The set of even-indexed Fibonacci numbers is definable in $\langle \mathbb{Z}, +, -, 0, 1, <, f(x) = \lfloor \varphi x \rfloor \rangle$.

Enrich the language:

Add to the language a function $F(x) =$ the largest even Fibonacci number smaller than x.

Theorem

$$c^{M_2}_{[a,b]} = c^{M_1}_{[a,b]}$$

Whenever $M_1 \subseteq M_2$ *are models and* $a, b, c, d \in M_1$*.*

Proof.

To be explained to the audience. □

Axiom

If there is t such that $a < t < b$ and $[\varphi c] < [\varphi t] < [\varphi d]$ then there is $t \in (a, b)$ such that

$$[\varphi t] = \min\{[\varphi x] : [\varphi c] < [\varphi x] < [\varphi d]\} \qquad (1)$$

That is $[\varphi t]$ is the larger but the closest to $[\varphi c]$.

Observation

Whenever t is as in (1) then

$$t = a + c^M_{(a-c,b-c)}.$$

IMPORTANT: THE CONVERSE DOES NOT HOLD.

Theorem

Let $M_1 \subseteq M_2$ be models. If the following holds in M_2 then it does so in M_1:

$$\exists x \quad (a < x < b) \wedge ([\varphi c] < [\varphi x] < [\varphi c])$$

Proof.

If there is x as above, then there is x with the above property whose decimal is the closest to $[\varphi c]$. By Observation in the previous page (page 22) we have
$x = a + c^{M_2}_{(a-c,b-c)} = a + c^{M_1}_{(a-c,b-c)} \in M_1$. □

Corollary (with further justifications!)

$\langle \mathbb{Z}, +, -, 0, 1, <, f(x) = \lfloor \varphi x \rfloor \rangle$ *is model-complete (using this we can prove that it is also complete and decidable)*